 whizkiddeng@gmail.com

The decomposition of rational function:

For the general theory you can check the page behind. here we just give the examples.

Q1. $\int \frac{1}{x^2+3x-10}\,dx$ (1)

Pf: Observe that $x^2+3x-10$ can be factorized like:

$(x^2+3x-10) = (x-2)(x+5)$ (where 2, -5 are roots of $Q(x) = x^2+3x-10$)

so $\frac{1}{x^2+3x-10} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5)+B(x-2)}{(x-2)(x+5)}$

$\begin{cases} A+B=0 \\ 5A-2B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{7} \\ B=-\frac{1}{7} \end{cases}$

so (1) $= \frac{1}{7}\left(\int \frac{1}{x-2}dx - \int \frac{1}{x+5}dx\right) = \frac{1}{7}(\ln|x-2| - \ln|x+5|) + C$

Remark: for $Q(x) = x^2+ax+b$ if $\Delta = a^2-4b \geq 0$ which means we have roots for $Q(x)=0$, so we can do the factorization like above;

otherwise if $\Delta = a^2-4b<0$, we need to use the complete square method:

$Q(x) = x^2+ax+b = (x+\frac{a}{2})^2 + b - \frac{a^2}{4} > 0$ compare this to t^2+1, so $\int \frac{1}{t^2+1}dt = \arctan t + C$.

Q2: $\int \frac{2x+4}{x^2+3x-10}\,dx$ (2)

Pf: for such case the order of $P(x)=2x+4$ just 1 less than $Q(x) = x^2+3x-10$.

we first consider $Q'(x) = 2x+3$. And extract such term from $P(x)$:

$$(2) = \int \frac{2x+3}{x^2+3x-10}dx + \int \frac{1}{x^2+3x-10}dx$$

$$= \int \frac{Q'(x)}{Q(x)}dx + \int \frac{1}{x^2+3x-10}dx$$

$$= \ln|x^2+3x-10| + (1) \text{ (above)} + C.$$

Q3. $\int \frac{1}{(x-1)(x+1)^2} dx \quad (3)$

Pf: $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \quad (4)$

for $x-1$, $x+1$ both linear factor, so A, B, C just constants

$(4) = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$, compare the coefficients of both sides:

$$\begin{cases} A+B=0 & (x^2 \text{ term}) \\ 2A+C=0 & (x \text{ term}) \\ A-B-C=0 & (\text{constant term}) \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=-\frac{1}{2} \end{cases}$$

$(3) = \frac{1}{4}\int \frac{1}{x-1} - \frac{1}{4}\int \frac{1}{x+1} - \frac{1}{2}\int \frac{1}{(x+1)^2}$

$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}(x+1)^{-1} + C$

Q4. $\int \frac{1}{(x+1)(x^2+1)} dx \quad (5)$

Pf: $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ (for x^2+1 is second order factor)

$= \frac{Ax^2+Bx+A+C+Bx^2+Cx}{(x+1)(x^2+1)}$

so $\begin{cases} A+B=0 \\ B+C=0 \\ A+C=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=\frac{1}{2} \end{cases}$

$(5) = \frac{1}{2}\int \frac{1}{x+1} - \frac{1}{2}\int \frac{x-1}{x^2+1} dx$

$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\times\frac{1}{2}\int \frac{2x}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1} \quad (Q(x)=x^2+1,\ Q'(x)=2x)$

$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan x + C$

Q5. $\int \frac{1}{x+\sqrt{x^2+x+1}}\,dx$ (6)

Pf: take substitution $t = x+\sqrt{x^2+x+1}$

So $t-x = \sqrt{x^2+x+1} \Rightarrow (t-x)^2 = t^2-2tx+\cancel{x^2} = \cancel{x^2}+x+1$

$x = \frac{t^2-1}{1+2t}$, $dx = 2\frac{t^2+t+1}{(1+2t)^2}\,dt$

$(6) = 2\int \frac{1}{t}\cdot\frac{t^2+t+1}{(1+2t)^2}\,dt$. factorize $\frac{t^2+t+1}{t(1+2t)^2} = \frac{A}{t} + \frac{B}{(1+2t)} + \frac{C}{(1+2t)^2}$

$\Rightarrow \begin{cases} A = 1 \\ B = -\frac{3}{2} \\ C = -\frac{3}{2}. \end{cases}$

$(6) = 2\left(\int\frac{1}{t} - \frac{3}{2}\int\frac{1}{1+2t} - \frac{3}{2}\int\frac{1}{(1+2t)^2}\right)$

$= 2\left(\ln|t| - \frac{3}{4}\ln|1+2t| + \frac{3}{2}(1+2t)^{-1}\right) + C.$

replace $t = x+\sqrt{x^2+x+1}$ get the final result.

Remark: for $\sqrt{a}x \pm \sqrt{ax^2+bx+c}$ form $(a\geq 0)$, we can always use substitution like $t = \sqrt{a}x \pm \sqrt{ax^2+bx+c}$, that's called Euler transform.

Q6. $I = \int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx.$ (7)

Pf: From $\begin{cases} \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2},\ t=\tan\frac{x}{2} \\ \sin x = \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \\ dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}(1+t^2)\,dx \end{cases}$

we can transform the trigonometric function to rational function.

$(7) = \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}\,\frac{2}{1+t^2}\,dt$

$= \frac{1}{2}\int (t+2+\frac{1}{t})\,dt$

$= \frac{1}{2}\left(\frac{1}{2}t^2+2t+\ln|t|\right)+C$

back to x is ok.

If we try to use traditional method to solve it, we have to consider:

$$I = \int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx \;,\quad J = \int \frac{\cos x}{\sin x(1+\cos x)}\,dx$$

$1+\cos x = 2\cos^2\frac{x}{2}$

$$I + J = \int \frac{1+\cos x+\sin x}{\sin x\,(1+\cos x)}\,dx = \int \frac{1}{\sin x}dx + \int \frac{1}{1+\cos x}\,dx \rightarrow \int \frac{1}{2\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx \quad \text{done}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\sin x}{\sin^2 x}dx \rightarrow -\frac{d\cos x}{1-\cos^2 x} \qquad \tan\frac{x}{2}$$

$1-\cos x = 2\sin^2\frac{x}{2}$

$$I - J = \int \frac{1+\sin x-\cos x}{\sin x\,(1+\cos x)}\,dx = \int \frac{1}{1+\cos x} + \int \frac{1-\cos x}{\sin x(1+\cos x)}\,dx \rightarrow \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}\,dx$$

$\downarrow$ as above

$$\downarrow$$

$$-\int \frac{d\cos\frac{x}{2}}{\cos^3\frac{x}{2}} \rightarrow \frac{1}{2}\cos^{-2}\frac{x}{2} \quad \text{done.}$$

we need to use some tricks.

$$I_n = \frac{1}{n-1}\tan^{n-1}x - I_{n-2}.$$

(2) 若 $I(m,n)=\int\cos^m x\sin^n x\,\mathrm{d}x$，则当 $m+n\neq 0$ 时，

$$\begin{aligned} I(m,n) &= \frac{\cos^{m-1}x\sin^{n+1}x}{m+n}+\frac{m-1}{m+n}I(m-2,n)\\ &= -\frac{\cos^{m+1}x\sin^{n-1}x}{m+n}+\frac{n-1}{m+n}I(m,n-2),\\ & n,m=2,3,\cdots. \end{aligned}$$

5. 利用上题的递推公式计算：

(1) $\int\tan^3 x\,\mathrm{d}x$；　　(2) $\int\tan^4 x\,\mathrm{d}x$；

(3) $\int\cos^2 x\sin^4 x\,\mathrm{d}x$.

6. 导出下列不定积分对于正整数 n 的递推公式：

(1) $I_n=\int x^n\mathrm{e}^{kx}\mathrm{d}x$；　　(2) $I_n=\int(\ln x)^n\mathrm{d}x$；

(3) $I_n=\int(\arcsin x)^n\mathrm{d}x$；　　(4) $I_n=\int\mathrm{e}^{ax}\sin^n x\,\mathrm{d}x$.

7. 利用上题所得递推公式计算：

(1) $\int x^3\mathrm{e}^{2x}\mathrm{d}x$；　　(2) $\int(\ln x)^3\mathrm{d}x$；

(3) $\int(\arcsin x)^3\mathrm{d}x$；　　(4) $\int\mathrm{e}^{x}\sin^3 x\,\mathrm{d}x$.

§3 有理函数和可化为有理函数的不定积分

至此我们已经学得了一些最基本的积分方法.在此基础上,本节将讨论某些特殊类型的不定积分,这些不定积分无论怎样复杂,原则上都可按一定的步骤把它求出来.

一 有理函数的不定积分

有理函数是指由两个多项式函数的商所表示的函数,其一般形式为

$$R(x)=\frac{P(x)}{Q(x)}=\frac{\alpha_0x^n+\alpha_1x^{n-1}+\cdots+\alpha_n}{\beta_0x^m+\beta_1x^{m-1}+\cdots+\beta_m},\tag{1}$$

其中 n,m 为非负整数,$\alpha_0,\alpha_1,\cdots,\alpha_n$ 与 $\beta_0,\beta_1,\cdots,\beta_m$ 都是常数,且 $\alpha_0\neq0,\beta_0\neq0$.若 $m>n$,则称它为**真分式**;若 $m\leqslant n$,则称它为**假分式**.由多项式的除法可知,假分式总能化为一个多项式与一个真分式之和.由于多项式的不定积分是容易求得的,因此只需研究真分式的不定积分,故设(1)为一有理真分式.

根据代数知识,有理真分式必定可以表示成若干个部分分式之和(称为**部分**

分式分解).因而问题归结为求那些部分分式的不定积分.为此,先把怎样分解部分分式的步骤简述如下(可与后面的例1对照着做):

第一步 对分母 $Q(x)$在实系数内作标准分解:

$$Q(x)=(x-a_1)^{\lambda_1}\cdots(x-a_s)^{\lambda_s}(x^2+p_1x+q_1)^{\mu_1}\cdots(x^2+p_tx+q_t)^{\mu_t}, \tag{2}$$

其中 $\beta_0=1,\lambda_i,\mu_j(i=1,2,\cdots,s;j=1,2,\cdots,t)$均为自然数,而且

$$\sum_{i=1}^{s}\lambda_i+2\sum_{j=1}^{t}\mu_j=m;p_j^2-4q_j<0,j=1,2,\cdots,t.$$

第二步 根据分母的各个因式分别写出与之相应的部分分式:对于每个形如$(x-a)^k$ 的因式,它所对应的部分分式是

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_k}{(x-a)^k};$$

对每个形如$(x^2+px+q)^k$ 的因式,它所对应的部分分式是

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_kx+C_k}{(x^2+px+q)^k}.$$

把所有部分分式加起来,使之等于 $R(x)$.(至此,部分分式中的常数系数 A_i,B_i,C_i 尚为待定的.)

第三步 确定待定系数:一般方法是将所有部分分式通分相加,所得分式的分母即为原分母 $Q(x)$,而其分子亦应与原分子 $P(x)$恒等.于是,按同幂项系数必定相等,得到一组关于待定系数的线性方程,这组方程的解就是需要确定的系数.

例1 对 $R(x)=\dfrac{2x^4-x^3+4x^2+9x-10}{x^5+x^4-5x^3-2x^2+4x-8}$作部分分式分解.

解 按上述步骤依次执行如下:

$$\begin{aligned}Q(x)&=x^5+x^4-5x^3-2x^2+4x-8\\&=(x-2)(x+2)^2(x^2-x+1).\end{aligned}$$

部分分式分解的待定形式为

$$R(x)=\frac{A_0}{x-2}+\frac{A_1}{x+2}+\frac{A_2}{(x+2)^2}+\frac{Bx+C}{x^2-x+1}. \tag{3}$$

用 $Q(x)$乘上式两边,得一恒等式

$$\begin{aligned}2x^4-x^3+4x^2+9x-10&\equiv A_0(x+2)^2(x^2-x+1)\\&\quad+A_1(x-2)(x+2)(x^2-x+1)+A_2(x-2)(x^2-x+1)\\&\quad+(Bx+C)(x-2)(x+2)^2.\end{aligned} \tag{4}$$

Tutorial 11 Min, Jie jmin@math.cuhk.edu.hk

* Last time: ① use formula of $\sin x$, $\cos x$, $\tan x$, etc.
② integration by parts
③ integration by substitution
④ Recurrence method.

* Trigonometric substitution

$\int \frac{a}{1+x^2}\,dx$, use $x=\tan\theta$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\sqrt{a-bx^2}$ $x=\sqrt{\frac{a}{b}}\sin\theta$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Eg. $\int \frac{x^2}{1+x^2}\,dx$

$$= \int dx - \int \frac{1}{1+x^2}\,dx = \int dx - \int \frac{1}{1+\tan^2\theta}\,d\tan\theta \qquad \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$= \int dx - \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int dx - \int d\theta = x+\theta+C$$

$$= x+\arctan x + C$$

Eg. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

$$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,d\,3\sin\theta = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int (1-\cos 2\theta)\,d\theta = \frac{9}{2}\int d\theta - \frac{9}{4}\int \cos 2\theta\,d2\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C = \frac{9}{2}\arcsin\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + C$$

$\frac{x}{3}$

"

$2\sin\cos$

$3\ \frac{9}{26}\ \frac{9}{8}\ x\sqrt{-x^2}$

* Integration of rational functions

make $f(x) = \frac{p(x)}{q(x)} = \frac{a}{x+b} + \frac{cx}{x^2+d} + \frac{1}{x^2+e} + \cdots$

then integrate each term using previous methods.

Eg: $\int \frac{1}{(x+1)(x^2+1)} dx$

$= \frac{1}{2}\int \frac{1}{x+1} dx - \frac{1}{4}\int \frac{2x}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2+1} dx$

$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(1+x^2) + \frac{1}{2}\arctan x + C$

$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$\Leftrightarrow Ax^2 + A + Bx^2 + Bx + Cx + C = 1$

$\Leftrightarrow \begin{cases} A+B=0 \\ B+C=0 \\ A+C=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=\frac{1}{2} \end{cases}$

Eg. $\int \frac{x^2+5x+4}{x^4+5x^2+4} dx$

$= \frac{5}{6}\int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx - \frac{5}{6}\int \frac{2x}{x^2+4} dx$

$= \frac{5}{6}\ln(1+x^2) - \frac{5}{6}\ln(x^2+4) + \arctan x + C$

$\frac{x^2+5x+4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$

$\Leftrightarrow Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D$
$= x^2 + 5x + 4$

$\Leftrightarrow \begin{cases} A+C=0 \\ B+D=1 \\ 4A+C=5 \\ 4B+D=4 \end{cases} \Leftrightarrow \begin{cases} A=\frac{5}{3} \\ B=1 \\ C=-\frac{5}{3} \\ D=0 \end{cases}$

Exercises :

① $$\int \frac{dx}{x\sqrt{x^2+1}}$$ let $x=\tan\theta$ $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$= \int \frac{\sec^2 d\theta}{\tan\theta \ \sec\theta} = \int \frac{1}{\sin\theta} d\theta = \int \frac{\sin\theta}{1-\cos\theta} d\theta$$

$$= \int \frac{d\cos\theta}{1+\cos\theta} + \int \frac{d\cos\theta}{1-\cos\theta} = \ln(1+\cos\theta) + \ln(1-\cos\theta) + C$$

$$= \ln\left(1+\frac{1}{\sqrt{1+x^2}}\right) + \ln\left(1-\frac{1}{\sqrt{1+x^2}}\right) + C$$

② $$\int \frac{x^2}{1-x^2} dx$$

$$= \int \frac{1}{1-x^2} dx - \int dx = \int \frac{1}{1+x} dx + \int \frac{1}{1-x} d(1-x) - \int dx$$

$$= \ln|1+x| + \ln|1-x| - x + C$$

③ $$\int \frac{4-2x}{(x^2+1)(x-1)^2} dx$$

$$= \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx - \int \frac{2}{x-1} dx + \int \frac{1}{(x-1)^2} dx$$

$$= \ln(x^2+1) + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C$$

④ $$\int \frac{\arctan x}{x^2+1} dx$$ let $x=\tan\theta$

$$= \int \frac{\theta}{\sec^2\theta} \sec^2\theta \, d\theta = \frac{1}{2}\theta^2 + C = \frac{1}{2}(\arctan x)^2 + C$$

⑤ $$\int \frac{1}{\sqrt{x-1}+\sqrt{(x-1)^3}} dx$$

$$= \int \frac{1}{\sqrt{x-1}\,(1+x-1)} dx = \int \frac{1}{x\sqrt{x-1}} dx$$ let $t=\sqrt{x-1}$

$x=t^2+1$

$dx=2t\,dt$

$$= \int \frac{2t}{(1+t^2)t} dt = 2\int \frac{1}{1+t^2} dt$$

$$= 2\arctan t + C = 2\arctan\sqrt{x-1} + C$$

Tutorial 11

Topics : Indefinite Integration $\begin{cases} \cdot \text{ Trigonometric Substitutions} \\ \cdot \text{ Partial fractions} \end{cases}$

Q1) Evaluate the integrals by trigonometic substitutions.

a) $\int e^{x} \sin x \, dx$ b) $\int \frac{1}{1-\sin x} \, dx$ c) $\int \sqrt{\frac{1+x}{1-x}} \, dx$

Q2) Evaluate the integrals by pertial fractions.

a) $\int \frac{x^3 + x^2 + x + 1}{x^3 - 3x^2 + 2x} \, dx$ b) $\int \frac{x^2 - 29x + 5}{(x-4)^2 (x^2+3)} \, dx$

Q3) Evaluate the integral. $\int |x|^3 + x^3 \, dx$

Recall:

- differentials of trigonometric functions

$d\sin x = \cos x\, dx$, $d\tan x = \sec^2 x\, dx$, $d\sec x = \tan x \sec x\, dx$,

$d\cos x = -\sin x\, dx$, $d\cot x = -\csc^2 x\, dx$, $d\csc x = -\cot x \csc x\, dx$

- trigonometric identities

$$1 = \sin^2 x + \cos^2 x$$

$$1 = \sec^2 x - \tan^2 x$$

$$1 = \csc^2 x - \cot^2 x$$

partial fractions

e.g. $$\frac{1}{(x-a)^2(x-b)(x^2+cx+d)}$$ where $a \neq b$, x^2+cx+d be irreducible.

$$= \frac{A}{(x-a)^2} + \frac{B}{x-a} + \frac{C}{x-b} + \frac{Dx+E}{x^2+cx+d}$$

To solve A, B, C, D, E by comparing the coefficients of

$$1 \equiv A(x-b)(x^2+cx+d) + B(x-a)(x-b)(x^2+cx+d)$$
$$+ C(x-a)^2(x^2+cx+d) + (Dx+E)(x-a)^2(x-b).$$

$\underline{\text{Sol}^n}$

Q1(a)

$$\int e^x \sin x\, dx = \int \sin x\, de^x = e^x \sin x - \int e^x \cos x\, dx$$

$$= e^x \sin x - \int \cos x\, de^x = e^x(\sin x - \cos x) + \int e^x\, d\cos x$$

$$= e^x(\sin x - \cos x) - \int e^x \sin x\, dx$$

$$\Rightarrow \int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C \qquad \exists\, C \in \mathbb{R}.$$

1b)

$$\int \frac{1}{1-\sin x}\, dx = \int \frac{1+\sin x}{1-\sin^2 x}\, dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x + \sec x \tan x\, dx = \int d(\tan x + \sec x)$$

$$= \tan x + \sec x + C \qquad \exists\, C \in \mathbb{R}.$$

(c) $\displaystyle \int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx$

$$= \int \frac{1+\sin t}{\sqrt{1-\sin^2 t}}\, d\sin t \qquad \text{by sub } x = \sin t$$

$$= \int (1+\sin t)\left(\frac{1}{\cos t}\right)(\cos t\, dt)$$

$$= \int 1 + \sin t\, dt$$

$$= t - \cos t + C \qquad \exists\, C \in \mathbb{R}.$$

$$= \sin^{-1} x - \cos(\sin^{-1} x) + C \quad //$$

2a) Consider

$$\frac{x^3+x^2+x+1}{x^3-3x^2+2x} = \frac{(x^3-3x^2+2x)+(4x^2-x+1)}{x^3-3x^2+2x} = 1+\frac{4x^2-x+1}{x(x-1)(x-2)}$$

$$= 1+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}$$

Solve A, B, C by comparing coefficient of

$$x^3+x^2+x+1 = (x)(x-1)(x-2) + A(x-1)(x-2) + B(x)(x-2) + C(x)(x-1)$$

$$\Rightarrow \quad A = 1/2, \quad B = -4, \quad C = 15/2$$

Hence $$\int \frac{x^3+x^2+x+1}{x^3-3x^2+2x}\,dx = \int 1 + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) + (-4)\left(\frac{1}{x-1}\right) + \left(\frac{15}{2}\right)\left(\frac{1}{x-2}\right)dx$$

$$= x + \frac{1}{2}\ln|x| + (-4)\ln|x-1| + \frac{15}{2}\ln|x-2| + C$$

$\exists C \in \mathbb{R}$ //

2b) Consider $\dfrac{x^2 - 29x + 5}{(x-4)^2(x^2+3)} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{Cx}{x^2+3} + \dfrac{D}{x^2+3}$

By comparing terms in

$$x^2 - 29x + 5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$$

$$= (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12$$

Hence $A = 1$, $B = -5$, $C = -1$, $D = 2$.

Hence $\displaystyle\int \frac{x - 29x + 5}{(x-4)^2(x^2+3)}\,dx$

substitute

$x = \sqrt{3}\tan y$

$$= \int \left[\frac{1}{x-4} + \frac{-5}{(x-4)^2} + \frac{-x}{x^2+3} + \frac{2}{x^2+3}\right] dx$$

$$= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2}\ln|x^2+3| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$\exists\ C \in \mathbb{R}$.

Q3) for $x \geq 0$, $|x^3| = x^3$

$$\Rightarrow \int |x^3| + x^3 \, dx = \int 2x^3 \, dx = \frac{2x^4}{4} + c = \frac{x^4}{2} + c \quad \exists c \in \mathbb{R}.$$

for $x \leq 0$, $|x^3| = -x^3$

$$\Rightarrow \int |x^3| + x^3 \, dx = \int -x^3 + x^3 \, dx = \int 0 \, dx = c' \quad \exists c' \in \mathbb{R}.$$

since at $x = 0$

$$c'\Big|_{x=0} = \frac{x^4}{2} + c\Big|_{x=0} \Rightarrow c = c'$$

Overall,

$$\int |x^3| + x^3 \, dx = \begin{cases} \frac{x^4}{2} + c, & x \geq 0 \\ c, & x \leq 0 \end{cases}$$

//